Mystery of Beal Equation:

$a^x + b^y = c^z$

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ABSTRACT

The proof of Beal equation mystery has been published in this journal vide Vol-4 of Aug-edition, 2013. Subsequent developments of the original theory were published in October & November editions. Due to phase-wise developments the overall true picture of Beal equation mystery was not reflected in the first manuscript. Now the picture is crystal clear and for the ease of apprehension I feel an urge to prepare a fresh manuscript consisting all the topics of three editions and incorporating them in a proper order and eliminating some insignificant/irrelevant topics.
This paper contains the same proof of Beal Equation Mystery i.e. without common factor among all the bases a Beal equation cannot exist and then the theory behind the formation of Beal Equation. It is, as I believe more scientific, more easy to understand and more presentable.

Keywords

Beal equation, N-equation & NZ-equation, N_d operation & N_s operation, Mixed Zygote form & odd Zygote form,

1. Introduction

$\mathbf{a^x + b^y = c^z}$, where a, b, c, x, y, z all are of positive integers & x, y, z > 2 is known as Beal Equation .
If this Beal Equation exists there must be a common factor among all the bases a, b, c e.g. $2^3 + 2^3 = 2^4$,
$7^6 + 7^7 = 98^3$, $19^4 + 38^3 = 57^3$ etc.

The proof of this mystery is still unknown to all mathematical communities. I believe that I have been able to give a proof in favor of this mystery.

The proof is mainly based on the properties of Pythagorean equation $a^2 + b^2 = c^2$ where a, b, c are of positive integers without any common factor among them.

The proof clearly shows that only one element of a, b, c can produce powers beyond two like 3, 4, 5, 6, ….. so as to receive three types of equations $a^n + b^2 = c^2$ or $a^2 + b^{2n} = c^2$ or $a^2 + b^2 = c^n$ under N-equation barring some cases where two elements can also produce power such as $1 + 2^3 = 3^2$, $7^2 + 2^5 = 3^4$etc. which fall under NZ-equation.

For N-equation the LH odd element produces power by N_d operation among mixed zygote expressions i.e. mixed with odd & even elements.
LH even element produces power by N_d operation among odd zygote expressions i.e. mixed with only odd elements.
RH odd element produces power by N_s operation among mixed zygote expressions.
Any two of these three operations or all the three are not possible to run simultaneously.
So in N-equation only one element can raise its power beyond two. But in NZ-equation two elements can produce power.

Apart from the proof that without common factor among the bases a Beal Eq. cannot exist, it also shows how Beal equations are formed and there cannot be any common factor among the exponents of Beal Equation.

2. Natural Equation or simply N-equation.

$a^2 + b^2 = c^2$ where the elements a,b,c all are of positive integers is said to be Natural equation provided its comparable equation i.e. $(\alpha^2 - \beta^2)^2 + (2\alpha\beta)^2 = (\alpha^2 + \beta^2)^2$ …………. Eq(A) has the property that α, β must be of positive integers. (a, b, c) is said to be a prime set when there is no common factor among a, b, c & said to be composite set when there lies a common factor.
Now from the property $(a\mu)^2 + (b\mu)^2 = (c\mu)^2$ we can say that any prime set (a,b,c) can produce infinite number of composite sets (aμ,bμ,cμ) where μ is any positive integer. But our concerned area is only for prime sets.
Now $a^2 + b^2 = c^2$ can be of three types
i) $(e_1)^2 + (e_2)^2 = (e_3)^2$ ii) $(o_1)^2 + (o_2)^2 = (o_3)^2$ iii) $(o_1)^2 + (e_1)^2 = (o_2)^2$
where e & o denote even and odd numbers respectively.
Case i) cannot be accepted as it is a composite set.

Case ii) cannot be accepted as $(o_1)^2 + (o_2)^2 = (2x - 1)^2 + (2y - 1)^2 = 2(2x^2 + 2y^2 - 2x - 2y + 1) =$ 2(an odd number) where x, y are positive integers, which cannot be a square quantity.
Case iii) is accepted and these can be of two kinds.

3. Natural equation of 1st kind and 2nd kind.

To maintain the ascending order i.e. $a < b < c$, 1st kind is defined as odd < even < odd
$\Rightarrow 2\alpha\beta > \alpha^2 - \beta^2 \Rightarrow (\alpha/\beta) < \sqrt{2} + 1$ and 2nd kind is just its reverse.
From Eq.(A), it is obvious α, β are the combination of odd and even.
For 1st kind $c - b = (\alpha - \beta)^2 = (\text{an odd no.})^2 = k$ say, where k can be said as natural constant.
Also $c - a = 2\beta^2 = 2(\text{an integer})^2$
Similarly for 2nd kind $k = c - b = 2(\text{an integer})^2$ & $c - a = (\text{an odd no.})^2$.

3.1. Natural equation of 1st kind in functional form.

Here, $[b + (2x - 1)^2 - 2y^2]^2 + b^2 = [b + (2x - 1)^2]^2$
where, $2y^2$ is just greater than $(2x - 1)^2$ by an integer value.
or, $b^2 - b.4y^2 + 4y^4 - 4y^2(2x - 1)^2 = 0$ or, $b = 2y^2 \pm 2y(2x - 1)$
or, $b = 2y^2 + 4xy - 2y$, neglecting (-) sign.
$\therefore\ a = 4x^2 + 4xy - 4x - 2y + 1$ & $c = 4x^2 + 2y^2 + 4xy - 4x - 2y + 1$
$\therefore$ leading functional set $(a,b,c) = [Lf_k(x), L\phi_k(x), L\psi_k(x)] =$
$[4x^2 + 4xy - 4x - 2y + 1,\ 2y^2 + 4xy - 2y,\ 4x^2 + 2y^2 + 4xy - 4x - 2y + 1]$
$\therefore$ for k = 1, put x = 1 & y = 1, to get the leading prime set (3,4,5)
for k = 9, put x = 2 & y = 3, to get the leading set (27,36,45)
for k = 25, put x = 3 & y = 4, to get the leading set (65,72,97)
for k = 49, put x = 4 & y = 5, to get the leading set (119,120,169) & so on

For a particular value of k we can change the functional expression so as to start the variable with one.
Say for k = 9 i.e. for x = 2, y starts from 3 i.e. $(6y + 9)^2 + (2y^2 + 6y)^2 = (2y^2 + 6y + 9)^2$ where $y \geq 3$
As 'a' is always linear & b, c are always quadratic expressions, say $a = Ax + B$ & $b = Cx^2 + Dx + E$
Obtain first three sets by putting y = 3, 4, 5 & they are (27, 36, 45), (33, 56, 65) & (39, 80, 89)
For x = 1, A + B = 27 & C + D + E = 36
For x = 2, 2A + B = 33 & 4C + 2D + E = 56
For x = 3, 9C + 3D + E = 80
Solving them we get $(6x + 21)^2 + (2x^2 + 14x + 20)^2 = (2x^2 + 14x + 29)^2$ where x = 1,2,3, ……..

3.2. Natural equation of 2nd kind in functional form.

Here, $k = 2x^2, \therefore\ (b + 2x^2 - y^2)^2 + b^2 = (b + 2x^2)^2$ where, y is an odd number so that y^2 is just greater than $2x^2$ by an integer value.
$\therefore\ b^2 - 2by^2 + (y^2 - 4x^2)y^2 = 0$ or, $b = y^2 \pm y\sqrt{(y^2 - y^2 + 4x^2)}$. Neglect (-)sign.
$\therefore\ b = y^2 + 2xy,\ a = y^2 + 2xy + 2x^2 - y^2 = 2x^2 + 2xy$ & $c = y^2 + 2xy + 2x^2$
$\therefore$ leading functional set $(a,b,c) = [Lf_k(x), L\phi_k(x), L\psi_k(x)] = [2x^2 + 2xy,\ y^2 + 2xy,\ y^2 + 2xy + 2x^2]$
$\therefore$ for k = 2, put x = 1, y = 3 to get the leading prime set (8,15,17)
for k = 8, put x = 2, y = 3 to get the leading set (20,21,29)
for k = 18, put x = 3, y = 5 to get the leading set (48,55,73)
for k = 32, put x = 4, y = 7 to get the leading set (88,105,137) & so on.

Here also, For a particular value of k we can change the functional expression so as to start the variable with one.
Followings are the few examples of N-equation in functional form.

1st kind. For k = 1, $(2x + 1)^2 + (2x^2 + 2x)^2 = (2x^2 + 2x + 1)^2$
For k = 9, $(6x + 21)^2 + (2x^2 + 14x + 20)^2 = (2x^2 + 14x + 29)^2$

For k = 25, $(10x + 55)^2 + (2x^2 + 22x + 48)^2 = (2x^2 + 22x + 73)^2$
For k = 49, $(14x + 105)^2 + (2x^2 + 30x + 88)^2 = (2x^2 + 30x + 137)^2$
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2nd kind. For k = 2, $(4x + 4)^2 + (4x^2 + 8x + 3)^2 = (4x^2 + 8x + 5)^2$
For k = 8, $(8x + 12)^2 + (4x^2 + 12x + 5)^2 = (4x^2 + 12x + 13)^2$
For k = 18, $(12x + 36)^2 + (4x^2 + 24x + 27)^2 = (4x^2 + 24x + 45)^2$
For k = 32, $(16x + 72)^2 + (4x^2 + 36x + 65)^2 = (4x^2 + 36x + 97)^2$
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Note:
For k=1&2, it will produce only prime sets, but in other cases some composite sets will appear intermittently as because the nature of 'k'in the relation $(a\mu)^2 + (b\mu)^2 = (c\mu)^2$, where μ = (an odd no,)^2 remains unaltered. In general, For $k = 2^n$ where n = 0,1,3,5,……., N-Eq produces only prime sets.

Coefficients of x^2 for 1st kind & 2nd kind are respectively 2&4. If it is taken into consideration then with the help of first two leading sets we can find out the functional sets of all leading sets. To obtain the first two leading sets we can adopt the following two simple methods.

Say, $k = 25 = 5^2$. Now, if the leading set be (a,b,c) then c = b + 25,
$a = c - 2.4^2$ [as 2.4^2 is just greater than 5^2] $\therefore$ a = b – 7.
$\therefore (b-7)^2 + b^2 = (b+25)^2$ or, b = 72. $\therefore$ 1st set is (65,72,97).
Similarly, for the 2nd set $a = c - 2.5^2$. [obviously, it will be a composite set]
$\therefore (b-25)^2 + b^2 = (b+25)^2$. or, b = 100. $\therefore$ 2nd set is (75,100,125).

Say, k = 18. $\therefore k = 2.3^2$.
Now, if the leading set be (a,b,c), then c = b + 18, $a = c - 5^2 = b - 7$.
$\therefore (b-7)^2 + b^2 = (b+18)^2$ or, b = 55 $\therefore$ a = 48 & c = 73 & 1st set is (48,55,73)
Similarly, for the 2nd set $a = c - 7^2$. $\therefore (b-31)^2 + b^2 = (b+18)^2$ or, b = 91
$\therefore$ a = 60 & c = 109 $\therefore$ 2nd set is (60,91,109).

4. Natural equation in Mixed Zygote form.

In Mixed Zygote form, a N-equation can be written as,

1st kind

$\{(y+2x-1)^2 - (y)^2\}^2 + \{2.y.(y+2x-1)\}^2 = \{(y+2x-1)^2 + (y)^2\}^2$ [as leading set]
or, $[\{f(x,y)\}^2 - \{\phi(y)\}^2]^2 + [2.f(x,y).\phi(y)]^2 = [\{f(x,y)\}^2 + \{\phi(y)\}^2]^2$
where, for a particular value of k, $\{f(x,y) - \phi(y)\}$ is constant.

2nd kind

$\{2.x.(x+y)\}^2 + \{(x+y)^2 - (x)^2\}^2 = \{(x+y)^2 + (x)^2\}^2$. [as leading set]
or, $[2.f(x,y).\phi(x)]^2 + [\{f(x,y)\}^2 - \{\phi(x)\}^2]^2 = [\{f(x,y)\}^2 + \{\phi(x)\}^2]^2$.
where, for a particular value of k, $\{\phi(x)\}$ is constant.

4.1. Example Chart of N-equation in Zygote form.

1st kind for k = 1, $(2^2 - 1^2)^2 + (2.2.1)^2 = (2^2 + 1^2)^2$.
$(3^2 - 2^2)^2 + (2.3.2)^2 = (3^2 + 2^2)^2$.
$(4^2 - 3^2)^2 + (2.4.3)^2 = (4^2 + 3^2)^2$.
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for k = 9, $(6^2 - 3^2)^2 + (2.6.3)^2 = (6^2 + 3^2)^2$.
$(7^2 - 4^2)^2 + (2.7.4)^2 = (7^2 + 4^2)^2$.

$(8^2 - 5^2)^2 + (2.8.5)^2 = (8^2 + 5^2)^2$.

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for k = 25, $(9^2 - 4^2)^2 + (2.9.4)^2 = (9^2 + 4^2)^2$.
$(10^2 - 5^2)^2 + (2.10.5)^2 = (10^2 + 5^2)^2$.
$(11^2 - 6^2)^2 + (2.11.6)^2 = (11^2 + 6^2)^2$.

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2nd kind for k = 2, $(2.4.1)^2 + (4^2 - 1^2)^2 = (4^2 + 1^2)^2$.
$(2.6.1)^2 + (6^2 - 1^2)^2 = (6^2 + 1^2)^2$.
$(2.8.1)^2 + (8^2 - 1^2)^2 = (8^2 + 1^2)^2$.

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for k = 8, $(2.5.2)^2 + (5^2 - 2^2)^2 = (5^2 + 2^2)^2$.
$(2.7.2)^2 + (7^2 - 2^2)^2 = (7^2 + 2^2)^2$.
$(2.9.2)^2 + (9^2 - 2^2)^2 = (9^2 + 2^2)^2$.

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for k = 18, $(2.8.3)^2 + (8^2 - 3^2)^2 = (8^2 + 3^2)^2$.
$(2.10.3)^2 + (10^2 - 3^2)^2 = (10^2 + 3^2)^2$.
$(2.12.3)^2 + (12^2 - 3^2)^2 = (12^2 + 3^2)^2$.

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5. Two important operations.

5.1 N_s operation

N_s operation is defined as $(a_1^2 + b_1^2)(a_2^2 + b_2^2) = (a_1b_2 \pm a_2b_1)^2 + (a_1a_2 -/+ b_1b_2)^2$
e.g. $65 = 5.13 = (2^2 + 1^2)(2^2 + 3^2) = (2.3 \pm 2.1)^2 + (2.2 -/+ 3.1)^2 = 8^2 + 1^2$ or, $4^2 + 7^2$.

5.2 N_d operation

N_d operation is defined as $(a_1^2 - b_1^2)(a_2^2 - b_2^2) = (a_1a_2 \pm b_1b_2)^2 - (a_1b_2 \pm a_2b_1)^2$
e.g. $35 = 5.7 = (3^2 - 2^2)(4^2 - 3^2) = (3.4 \pm 2.3)^2 - (3.3 \pm 4.2)^2 = 18^2 - 17^2$ or, $6^2 - 1^2$.

6. Power Characteristics of three elements of a N-equation $a^2 + b^2 = c^2$.

Here we consider a is L H odd element, b is L H even element & c is R H odd element.
Its comparable equation is $(\alpha^2 - \beta^2)^2 + (2\alpha\beta)^2 = (\alpha^2 + \beta^2)^2$ …………. Eq(A)
Here α, β can be said as its mixed zygote elements. $(\alpha^2 - \beta^2)$ & $(\alpha^2 + \beta^2)$ can be said as mixed zygote expressions which are conjugate to each other.

6.1 How the element 'a' produces power

a produces power from 2 to 3 by virtue of N_d operation in between $(\alpha^2 - \beta^2)$ & $(\alpha^2 - \beta^2)^2$
i.e. $(\alpha^2 - \beta^2)$ & $\{(\alpha^2 + \beta^2)^2 - (2\alpha\beta)^2\}$ on multiplication we get
$\{(\alpha^3 + \alpha\beta^2) \pm (2\alpha\beta^2)\}^2 - \{(2\alpha^2\beta) \pm (\alpha^2\beta + \beta^3)\}^2$
$\Rightarrow (\alpha^3 + 3\alpha\beta^2)^2 - (3\alpha^2\beta + \beta^3)^2$ or, $\{\alpha(\alpha^2 - \beta^2)\}^2 - \{\beta(\alpha^2 - \beta^2)\}^2$ 2nd one can be neglected as it is a composite set.
$\Rightarrow (\alpha^2 - \beta^2)^3 = (\alpha^3 + 3\alpha\beta^2)^2 - (3\alpha^2\beta + \beta^3)^2$
Again by repetitive multiplication of $(\alpha^2 - \beta^2)$ on both sides we get

$(\alpha^2 - \beta^2)^n + \{ {}^nc_1\alpha^{n-1}\beta + {}^nc_3\alpha^{n-3}\beta^3 + \ldots\ldots \}^2 = \{\alpha^n + {}^nc_2\alpha^{n-2}\beta^2 + \ldots\ldots\}^2$ ..……(B)

Note: for n = odd integer it will always produce a relation like $a^{2n+1} + b^2 = c^2$ where b is odd & c is even provided zygote expression of 'a' is in the form of $(\text{even})^2 - (\text{odd})^2$

6.2 How the element 'c' produces power

c produces power from 2 to 3 by virtue of N_s operation in between $(\alpha^2 + \beta^2)$ & $(\alpha^2 + \beta^2)^2$
i.e. $(\alpha^2 + \beta^2)$ & $\{(\alpha^2 - \beta^2)^2 + (2\alpha\beta)^2\}$ on multiplication we get
$\{(2\alpha^2\beta) \pm (\alpha^2\beta - \beta^3)\}^2 + \{(\alpha^3 - \alpha\beta^2) -/+ (2\alpha\beta^2)\}^2$
$\Rightarrow (3\alpha^2\beta - \beta^3)^2 + (\alpha^3 - 3\alpha\beta^2)^2$ or, $\{\beta(\alpha^2 + \beta^2)\}^2 + \{\alpha(\alpha^2 + \beta^2)\}^2$ 2nd one can be neglected as it is a composite set.
$\Rightarrow (\alpha^2 + \beta^2)^3 = (3\alpha^2\beta - \beta^3)^2 + (\alpha^3 - 3\alpha\beta^2)^2$
Again by repetitive multiplication of $(\alpha^2 + \beta^2)$ on both sides we get
$\{\alpha^n - {}^nc_2\alpha^{n-2}\beta^2 + \ldots\}^2 + \{{}^nc_1\alpha^{n-1}\beta - {}^nc_3\alpha^{n-3}\beta^3 + \ldots\}^2 = (\alpha^2 + \beta^2)^n$ ……..(C)

6.3 How the element 'b' produces power

Earlier we discussed the Mixed Zygote form of N-equation. Let us now introduce another form i.e. 'Odd zygote form' of N-equation.
We have $c + b = d_1^2$ & $c - b = d_2^2$. $\therefore$ $c = (d_1^2 + d_2^2)/2$ & $b = (d_1^2 - d_2^2)/2$
& $a = \sqrt{(c^2 - b^2)} = \sqrt{\{(c + b)(c - b)\}} = d_1d_2$.

$\Rightarrow \{(d_1^2 - d_2^2)/2\}^2 + (d_1d_2)^2 = \{(d_1^2 + d_2^2)/2\}^2$ Here, d_1 & d_2 both are odd.

The example chart can be given below.

1st kind

For k = 1,
$(1.3)^2 + \{(3^2 - 1^2)/2\}^2 = \{(3^2 + 1^2)/2\}^2$
$(1.5)^2 + \{(5^2 - 1^2)/2\}^2 = \{(5^2 + 1^2)/2\}^2$
$(1.7)^2 + \{(7^2 - 1^2)/2\}^2 = \{(7^2 + 1^2)/2\}^2$
$(1.9)^2 + \{(9^2 - 1^2)/2\}^2 = \{(9^2 + 1^2)/2\}^2$
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For k = 9,
$(3.9)^2 + \{(9^2 - 3^2)/2\}^2 = \{(9^2 + 3^2)/2\}^2$
$(3.11)^2 + \{(11^2 - 3^2)/2\}^2 = \{(11^2 + 3^2)/2\}^2$
$(3.13)^2 + \{(13^2 - 3^2)/2\}^2 = \{(13^2 + 3^2)/2\}^2$
$(3.15)^2 + \{(15^2 - 3^2)/2\}^2 = \{(15^2 + 3^2)/2\}^2$
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For k = 25,
$(5.13)^2 + \{(13^2 - 5^2)/2\}^2 = \{(13^2 + 5^2)/2\}^2$
$(5.15)^2 + \{(15^2 - 5^2)/2\}^2 = \{(15^2 + 5^2)/2\}^2$
$(5.17)^2 + \{(17^2 - 5^2)/2\}^2 = \{(17^2 + 5^2)/2\}^2$
$(5.19)^2 + \{(19^2 - 5^2)/2\}^2 = \{(19^2 + 5^2)/2\}^2$
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2nd kind.

For k = 2,
$\{(5^2 - 3^2)/2\}^2 + (5.3)^2 = \{(5^2 + 3^2)/2\}^2$
$\{(7^2 - 5^2)/2\}^2 + (7.5)^2 = \{(7^2 + 5^2)/2\}^2$
$\{(9^2 - 7^2)/2\}^2 + (9.7)^2 = \{(9^2 + 7^2)/2\}^2$
$\{(11^2 - 9^2)/2\}^2 + (11.9)^2 = \{(11^2 + 9^2)/2\}^2$
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For k = 8,
$\{(7^2 - 3^2)/2\}^2 + (7.3)^2 = \{(7^2 + 3^2)/2\}^2$
$\{(9^2 - 5^2)/2\}^2 + (9.5)^2 = \{(9^2 + 5^2)/2\}^2$
$\{(11^2 - 7^2)/2\}^2 + (11.7)^2 = \{(11^2 + 7^2)/2\}^2$
$\{(13^2 - 9^2)/2\}^2 + (13.9)^2 = \{(13^2 + 9^2)/2\}^2$
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For k = 18,
$\{(11^2 - 5^2)/2\}^2 + (11.5)^2 = \{(11^2 + 5^2)/2\}^2$
$\{(13^2 - 7^2)/2\}^2 + (13.7)^2 = \{(13^2 + 7^2)/2\}^2$
$\{(15^2 - 9^2)/2\}^2 + (15.9)^2 = \{(15^2 + 9^2)/2\}^2$
$\{(17^2 - 11^2)/2\}^2 + (17.11)^2 = \{(17^2 + 11^2)/2\}^2$
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Here, b^2 is expressible in the form of $\alpha_1^2 - \beta_1^2$ whereas b is expressible in the form of $(\alpha_1^2 - \beta_1^2)/2$ where α, β both are odd. So by N_d operation we cannot receive a relation like $b^3 = \alpha^2 - \beta^2$ But for b^4 by N_d operation we can always get a relation $(b.b/2)^2 = p^2 - q^2$ where p, q both are odd.
Hence,(any even no.)$^{\text{any odd no.}}$ cannot be a term of N-equation. It is under NZ-equation.

So it is observed that 'a' produces power by N_d operation among mixed zygote expressions i.e. mixed with odd & even elements.
'b' produces power by N_d operation among odd zygote expressions i.e. mixed with only odd elements.
'c' produces power by N_s operation among mixed zygote expressions.
Any two of these three operations or all the three are not possible to run simultaneously.
So in N-equation only one element can raise its power beyond two.

The general form of N-equation where even element (b) is in power form by continuous applications of N_d operations over b^2 & b^2,can be written as

$(b^n)^2 + ({}^nc_1c^{n-1}a + {}^nc_3c^{n-3}a^3 + \ldots\ldots)^2 = (c^n + {}^nc_2c^{n-2}a^2 + \ldots..)^2$
Which is a composite set with common factor 2^{n-1}.

$\Rightarrow (b^n/2^{n-1})^2 + (d_1)^2 = (d_2)^2$, where obviously, d_1 & d_2 are odd.
Say, $b = 2^m.\alpha^p$ where α is odd.
$\Rightarrow \{2^{n(m-1)+1}.\alpha^{pn}\}^2 + (d_1)^2 = (d_2)^2$, where GCF of n(m – 1) + 1 & pn > 1so as to receive the even element in power form.

7. The cases where two exponents are greater than two.

The N-eq. $a^2 + b^2 = c^2$, a < b < c can be defined as $(a_0^2 - b_0^2)^2 + (2a_0b_0)^2 = (a_0^2 + b_0^2)^2$, where a, b, c are N-elements & a_0, b_o are its zygote elements.
$(a_0^2 - b_0^2)$ & $(a_0^2 + b_0^2)$ are the corresponding zygote expressions conjugate to each other.

If the zygote elements are of positive integers, we can have the equation
$a^x + b^y = c^z$ where (a, b, c) is a prime set and if x > 2, y = 2 = z or, if y > 2, z = 2 = x or, if z > 2, x = 2 = y

If the zygote elements are of irrational nature i.e. in the form of (p ± q√r), we have the N-eq. renamed as N-eq. of irrational zygote elements or simply NZ -equation.
$\Rightarrow \{(p + q\sqrt{r})^2 - (p - q\sqrt{r})^2\}^2 + \{2(p + q\sqrt{r})(p - q\sqrt{r})\}^2 = \{(p + q\sqrt{r})^2 + (p - q\sqrt{r})^2\}^2$

or, $(4pq\sqrt{r})^2 + \{2(p^2 - q^2r)\}^2 = \{2(p^2 + q^2r)\}^2$
or, $\{p)^2 - (q\sqrt{r})^2\}^2 + \{2p.q\sqrt{r}\}^2 = \{\{p)^2 + (q\sqrt{r})^2\}^2$

Here also like N-eq. the RH term of NZ-equation can produce even power by virtue of N_s operation in between mother expression & self and odd power by same N_s operation in between mother & its zygote elements.
Similarly, corresponding LH term of NZ-eq. can produce power by virtue of N_d operation.

These two N_s & N_d operation cannot run simultaneously. Hence, only one element can produce power greater than two. But after N_d or N_s operations, the irrational element can produce power due to presence of √r factor. Here, if x, y > 2 then z = 2 & so on.
Let us write the N-eq. in power form:

$(\alpha^2 - \beta^2)^n + \{{}^nc_1\alpha^{n-1}\beta + {}^nc_3\alpha^{n-3}\beta^3 + \ldots.. \}^2 = \{\alpha^n + {}^nc_2\alpha^{n-2}\beta^2 + \ldots..\}^2$ ......(B)

$\{\alpha^n - {}^nc_2\alpha^{n-2}\beta^2 + \ldots..\}^2 + \{{}^nc_1\alpha^{n-1}\beta - {}^nc_3\alpha^{n-3}\beta^3 + \ldots\ldots. \}^2 = (\alpha^2 + \beta^2)^n$ ……..(C)

For NZ-eq. where α is integer & β is irrational Eq.(B) can be written in two ways.

$(\alpha^2 - \beta^2)^n + \{\beta f(\alpha, \beta, n)\}^2 = \{\alpha g(\alpha, \beta, n)\}^2$ when n is odd. ……. (B1)

$(\alpha^2 - \beta^2)^n + \{\alpha\beta f(\alpha, \beta, n)\}^2 = \{g(\alpha, \beta, n)\}^2$ when n is even. ……. (B2)

The integer element i.e. third one cannot produce power. If the irrational element i.e. second one produces power, $f(\alpha, \beta, n)$ must be in the form of β^{2m} in case of n is odd and in the form of $(\alpha\beta)^{2m}$ in case of n is even.

Similarly Eq.(C) can be written in two ways

$\{\alpha g_1(\alpha, \beta, n)\}^2 + \{\beta f_1(\alpha, \beta, n)\}^2 = (\alpha^2 + \beta^2)^n$ when n is odd ……(C1)
$\{g_1(\alpha, \beta, n)\}^2 + \{\alpha\beta f_1(\alpha, \beta, n)\}^2 = (\alpha^2 + \beta^2)^n$ when n is even ……(C2)

The integer element i.e. first one cannot produce power. If the irrational element i.e. second one produces power, $f_1(\alpha, \beta, n)$ must be in the form of β^{2m} in case of n is odd and in the form of $(\alpha\beta)^{2m}$ in case of n is even.

Example in favor of Eq.(C1)

for n = 3, $3\alpha^2 - \beta^2 = \beta^m$ or, $\beta^m + \beta^2 - 3\alpha^2 = 0$, where obviously, m is even & β is in the form of $q\sqrt{r}$, (q, r are odd) & there is no c.f. among α, p, q.
We have, $(\sqrt{3})^4 + (\sqrt{3})^2 = 3.2^2$ Hence, consider the equation,
$\{2^2 - (\sqrt{3})^2\}^2 + (2.2\sqrt{3})^2 = \{2^2 + (\sqrt{3})^2\}^2$ i.e. $1^2 + (4\sqrt{3})^2 = 7^2$.
By N_S operation in between 7^2 & 7 i.e. in between $1^2 + (4\sqrt{3})^2$ & $2^2 + (\sqrt{3})^2$ we get, $(8\sqrt{3} \pm \sqrt{3})^2 + (2 \mp 12)^2$ where one case is $3^5 + 10^2 = 7^3$.

Example in favor of Eq.(C2)

For n = 4, we have irrational element $\beta.4(\beta^2 - \alpha^2)$. Put $\alpha = 1$ & $\beta = \sqrt{2}$, we get$4(\beta^2 - \alpha^2) = 4 = (\sqrt{2})^4 = \beta^4$
Hence, consider the equation $\{(\sqrt{2})^2 - 1^2\}^2 + (2\sqrt{2})^2 = \{(\sqrt{2})^2 + 1^2\}^2$
or, $1^2 + (2\sqrt{2})^2 = 3^2$. Apply N_s operations in between $\{1^2 + (2\sqrt{2})^2\}$ & self.
$(2\sqrt{2} \pm 2\sqrt{2})^2 + (8 \mp 1)^2 = 3^2.3^2$ or, $2^5 + 7^2 = 3^4$ or, directly from Eq-(C) we get the same result.
On the same logic for n = 2, we get $1 + 2^3 = 3^2$.

As both the binomially expanded elements under Eq-(C) are sum of alternately (+) & (–), it will produce the relations of low value elements. But Eq-(B) will produce relations of high value elements.

Let us take the example of $17^3 + 2^7 = 71^2$. It is an example of low value elements. Here, if we proceed from 71 for N_s operation , by back calculation we can say,
$71 = [\sqrt{\{(71 + 8\sqrt{2})/2\}}]^2 + [\sqrt{\{(71 - 8\sqrt{2})/2\}}]^2 = p^2 + q^2$ (say)
Now applying Ns operation in between $(p^2 + q^2)$ & self we get the relation $17^3 + 2^7 = 71^2$.
We can proceed from the element 17 also. 17 must be expressed in the form of $p^2 - q^2$ where by successive N_d operations (3 times) on 17 we can get the same relation $17^3 + 2^7 = 71^2$, may be nature of p, q are different i.e. not in the form of p = s & $q = t\sqrt{u}$ where s, t, u are integers. For integer values of s, t, u N_d operations will produce relations of high value elements.

From Eq.(A) we can say $(p + q)^3 = 17 + 8\sqrt{2}$ & $(p - q)^3 = 17 - 8\sqrt{2}$
$\Rightarrow p = ½.[(17 + 8\sqrt{2})^{1/3} + (17 - 8\sqrt{2})^{1/3}$ & $q = ½.[(17 + 8\sqrt{2})^{1/3} - (17 - 8\sqrt{2})^{1/3}$

Computer generated some relations are given below.
$7^3 + 13^2 = 2^9$
$3^5 + 11^4 = 122^2$
$17^7 + 76271^3 = 21063928^2$
$1414^3 + 2213459^2 = 65^7$
$9262^3 + 15312283^2 = 113^7$
$43^8 + 96222^3 = 30042907^2$
$33^8 + 1549034^2 = 15613^3$ etc.
All can be explained in similar ways.

8. Theory behind the formation of Beal Equation.

The N-equation $a^2 + b^2 = c^2$ where a, c are odd elements & b is even elements can be redefined in power form as
$a^n + b^2 = c^2$ where a, b, c are powerless & n = 2, 3, 4, ……
$a^2 + b^2 = c^n$ where a, b, c are powerless & n = 2, 3, 4, ……
$a^2 + b^{2n} = c^2$ where a, b, c are powerless & n =1, 2, 3, 4, ……
Here, one cannot be the element of N-equation.
For NZ-equation two exponents can exceed beyond two. Exponent of one of the elements under N_d or N_s operations must be restricted to two. One can be element of NZ-equation.

A prime or composite set of three numbers (a, b, c) where a + b = c can produce a Beal equation after choosing a common multiplier provided any two numbers of a, b, c are in power form greater than two
and the powerless term must have at least one factor with power greater than 3, say $b = \beta^n\gamma = \beta^{n-r}.\beta^r\gamma$ where n > 3, n – r > 3. For the corresponding other factor $\alpha = \beta^r\gamma$, α^{p-1} will be common multiplier to produce Beal equation where GCF of (p – 1) with power of a, c separately & that of p with (n – r) will be greater than 3.
If a or b = 1, say a = 1, then for the powerless term (powerless means if $\beta = 1$, power of $\gamma \le 2$), say b, b^n will be CM where GCF of n with power of $c \ge 3$. It is true for a = 1 = b also.

In more simplified way we can say, for any two numbers A & B (> 1) which are prime to each other if it is found $A^m \pm B^n = \gamma^\alpha.\beta$ where $\alpha > 3$, then select a number p from 4,5,6 ….. , α such that GCF of p & α, (p – 1) & m, (p – 1) & n all are ≥ 3. If p exists then Beal eq. will exist with common multiplier β^{p-1}
Obviously, to produce Beal equation
m, n, α cannot be all prime.
For m ≠ n; m, n cannot be prime.
If α is prime; m or n cannot be prime.
As the consecutive nos. p & (p – 1) do not have any common factor in between them, hence for any Beal equation $A^x + B^y = C^z$, (x, y, z) is a prime set i.e. no common factor lies among x, y, z. In between two there can be a common factor. This implies all the powers cannot be even.

9. Few examples for the existence of $A^m \pm B^n = \gamma^\alpha.\beta$ where $\alpha > 3, \gamma \ge 1$

Any even number can be expressed as $N = 2^n.p$ where p is an odd integer & n is an integer known as degree of intensity of N i.e. D(N) = n.

Say, $N = A^m - B^n$ where A & B are two odd nos. ≥ 3 & prime to each other and m, n both are odd.
$\Rightarrow N = (1 + e_1)^m - (1 + e_2)^n$
After Binomial expansion, N = e_1(an odd integer) – e_2(an odd integer)
Say, $e_1 = 2^p.o_1$ & $e_2 = 2^q.o_2$ where o_1 & o_2 are odd integers.
⇒ if p ≠ q, $N = 2^\alpha$.(an odd no.) where α = Min (p, q) & to create Beal eq. α > 3
For p = q = α,
$N = 2^\alpha[(mo_1 - no_2) + ({}^mc_2.o_1^2 - {}^nc_2.o_2^2)2^\alpha + ({}^mc_3.o_1^3 - {}^nc_3.o_2^3).2^{2\alpha} + ……..]$
$= 2^{\alpha+\lambda}$.(an odd integer) where $\lambda_{min} = 1$. So α > 2
If $D(mo_1 - no_2) \ge 2$ then α > 1.

For m, n both are even,
$N = (me_1 - ne_2) + ({}^mc_2e_1^2 - {}^nc_2e_2^2) + ………$
⇒ D(N) > Min (p, q) + 1 ⇒ to create Beal eq. Min(p, q) > 2
Say m, n are combination of even & odd.
For p = q = α, $N = 2^\alpha$.(an odd integer). Hence, to create B eq. α > 3
For p ≠ q, Min(D(me_1), D(ne_2) > 3

Say, A & B are even & for any integer values of m, n (>3),
Say, $N = (1 + a^\alpha g)^m - (1 + a^\alpha h)^n = a^\alpha$(an even no.), after binomial expansion.
To create Beal eq. α > 3
Say, A is even & B is odd & m, n are any integers > 3
Say, $N = (1 + a^\alpha g)^m - (1+ 2^k a^\alpha h)^n = a^\alpha$(an odd no), after binomial expansion.

To create Beal eq. $\alpha > 3$

For a particular acceptable value of α there can be infinite no. of sets (m, n).

For a + b = c, any two number taken from each side must be in power form. Say, a & c (both > 1) are in extreme power form with bases a_1 & c_1. Then there must be a common factor among $(a_1 - 1)$, b, $(c_1 - 1)$ in the form of θ^k where $\theta = 1, 2, 3, 4, \ldots$ & $k > 3$ [$\theta = 1$ for $\gamma = 1$]. If not, it is to be understood that a, c are not prime to each other. Already there exists a common factor γ.

Let consider the case $N = A^m + B^n$ where A & B are any two odd integers (> 1) & m, n are any two integers ≥ 3
Say, $A = 1 + \gamma^\alpha.I_1$ & $B = 1 + \gamma^\alpha.I_2$
$\Rightarrow N = (1 + \gamma^\alpha.I_1)^m + (1 + \gamma^\alpha.I_2)^n$
$= 2[\, 1 + (mI_1 + nI_2)\gamma^\alpha + ({}^mc_2I_1^2 + {}^nc_2I_2^2)\gamma^{2\alpha} + \ldots\ldots.. \,]$
$= 2[\, 1 + \gamma^\alpha.P]$
Now $\gamma^\alpha.P$ & $(1 + \gamma^\alpha.p)$ two consecutive nos. cannot have a common factor γ^α.
$\Rightarrow N = 2$(an odd no.) & D(N) = 1
In spite of considering common factor γ^α result is free from γ^α. Hence, it's not capable of producing Beal eq. for $\gamma \neq 1$
Here the result 2(odd no.) i.e. $2.o_1^{p1}.o_2^{p2}$…..has the limitation $p_1, p_2, p_3, \ldots \leq 3$
(* subject to Proof)

Say, A is odd & B is even.
$N = (1 + e_1)^m + (1 + o_1)^n$ where $e_1 = \gamma^\alpha.e_2$ & $o_1 = \gamma^\alpha.o_2$, obviously γ is odd.
$= 2 + ({}^mc_1.e_2 + {}^nc_1.o_2)\gamma^\alpha + ({}^mc_2.e_2^2 + {}^nc_2.o_2^2)\gamma^{2\alpha} + \ldots\ldots.$
$= 2 + \gamma^\alpha.P \Rightarrow N$ does not have a c.f. γ^α as two consecutive odd nos. cannot have a common factor.
Hence, it is also not capable of producing any Beal eq. for $\gamma \neq 1$.

In the above all we have discussed considering the fact that A, B are prime to each other. If there is a common factor in between A & B, then it is always possible to have a relation like $A^m \pm B^n = \gamma^\alpha\beta$.

Conclusion

Whether my proof is correct or wrong it needs to be examined by an expert number-theorist and after all it should be accepted by all mathematical communities. The total nos. of solutions of $a^x + b^y = c^z$, where (a, b, c) is a prime set & any two of (x, y, z) > 2 & other = 2, seems to be finite. If it is so, how many? It needs further investigations.

Moreover, from N-eq. so many important things can be noticed such as:

a) if $(e_1^2 + o_1^2)(e_2^2 + o_2^2)$ produces a relation $e_3^2 + o_3^2 = e_4^2 + o_4^2$, then
$\text{Max}(e_3, e_4) + \text{Max}(o_3, o_4) = (e_1 + o_1)(e_2 + 0_2)$
$|\, \text{Max}(e_3, e_4) - \text{Max}(o_3, o_4) \,| = |(e_1 - o_1)(e_2 - 0_2)|$

b) if $(a_1^2 - b_1^2)(a_2^2 - b_2^2)$ produces a relation $a_3^2 - b_3^2 = a_4^2 - b_4^2$, then
$(a_1^2 + b_1^2)(a_2^2 + b_2^2)$ will produce a relation $a_3^2 + b_4^2 = a_4^2 + b_3^2$

c) For a N-equation $a^2 + b^2 = c^2$ where a & c are odd integers, the prime numbers excepting two can be divided into two types. Those who belong to 'c' as a prime factor or alone can be said as type-2 and the rest can be said as type-1 Obviously, type-2 prime nos. are distributed to all values of 'k' and remains present for a particular value of k uniquely i.e. $k = c - \text{Max}\{\, 2ab, (b^2 - a^2)\}$ whereas type-1 & type-2 both are belonging to 'a' under k = 1 as a prime factors or along.

If P_2 be a prime no. of type-2, then $P_2^{2^n}$, n = 0, 1, 2, 3, ...; can be expressed as $a^2 + b^2$ uniquely
All the prime numbers of type-1 & type-2 both with exponent 2^n, n = 0, 1, 2, 3, ….. can be expressed as difference of two square quantities of two consecutive nos. uniquely i.e. $\{(P + 1)/2\}^2 - \{(P - 1)/2\}^2$. But for type-1 prime number it is $(\text{even})^2 - (\text{odd})^2$ & for type-2 prime no. it is $(\text{odd})^2 - (\text{even})^2$.

Any composite no. whose all prime factors are of type-2 can be expressed as $(e)^2 + (o)^2$ & $(o)^2 - (e)^2$ but not uniquely.
Any composite no. whose at least one prime factor is of type-1 can be expressed as $(e)^2 - (0)^2$ but not uniquely. It cannot be expressed as $(e)^2 + (o)^2$

When N is found to be prime by digital analysis we can establish the following fact.

Digit of unit Place of N	Digit of 10^{th} Place of N	Remarks
1 or 9	even	'N' is of type-2
3 or 7	odd	'N' is of type-2
1 or 9	odd	'N' is of type-1
3 or 7	even	'N' is of type-1

$\Rightarrow D(P_1 - 1) = 1$& $D(P_2 - 1) > 1$ or, $(P_1 + 1)/2$ = even & $(P_2 + 1)/2$ = odd
* For any even number $N = 2^n.p$ where p is an odd integer, n is said to be degree of intensity & denoted by D.

d) With the help of N-equation it has been possible to analyze all the important aspects of Beal-equation. It also covers the proof of Fermat's Last Theorem i.e. $a^n + b^n = c^n$ where n is positive integer > 2, does not have any solution. Now we can put our attention over the fact $a^n + b^n = c^n + d^n$. For n = 2 all the relations are made available from the right hand element (c) of N-equation. Any composite number of c having at least two prime factors can produce such type of relations. But what happens when n > 2? It will be worthy to mention here that for n = 3, the minimum number having the property $a^3 + b^3 = c^3 + d^3$, was first noticed by great mathematician Sir Srinivasa Ramanujan i.e. $1^3 + 12^3 = 9^3 + 10^3 = 1729$ (Ramanujan number). But is there any relations among a, b, c, d or can all the relations be arranged in a systematic manner like N-eq?
From N or NZ equation one thing is clearly understood that $a^{2n} + b^{2n} = c^{2n} + d^{2n}$, for n > 1 does not have any solution. Because this type of relation is obtained only by N_s or N_d operation where two equal powers more than two is absurd.
So there lies ample of scopes for further development of N-eq. particularly in the field of prime numbers.

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About Author.

I am born and brought up in 'Kolkata', a small and well known city under the state 'West Bengal' of country 'India'. My date of birth is 12th July, 1958. By profession I am not a mathematician. My professional identity is that I am a simple graduate in civil engineering, working as a Senior Manager in Public Sector Unit 'Indian Oil Corporation Ltd', presently posted at Barauni Refinery, Bihar. I completed my graduation from 'Bengal Engineering College, Shibpur, Howrah, under Calcutta University' in 1980. But I feel immense pleasure to introduce myself as a fan of mathematics by my hobby. Service gives me motion but mathematics gives me emotion.

I first became familiar with this problem of Beal Equation through an article written by reporter Mr. Pathik Guha & published in a local newspaper about 13 years back. What the magical power does the problem have, I do not know, but since then it was sticking to my life just like my shadow. Whether my proof of Beal equation mystery will be accepted by all mathematical communities or not, only future can tell. But presently I am happy and grateful to IJSER that they have published my papers & brought my proof to the knowledge of all.